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On the equivalence between implicit regularization and constrained differential renormalization

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Abstract. Constrained differential renormalization (CDR) and the constrained version of implicit regularization are two regularization independent techniques that do not rely on dimensional continuation of the space-time. These two methods, which have rather distinct bases, have been successfully applied to several calculations, which show that they can be trusted as practical, symmetry invariant frameworks (gauge and supersymmetry included) in perturbative computations even beyond one-loop order. In this paper, we show the equivalence between these two methods at one-loop order. We show that the configuration space rules of CDR can be mapped into the momentum-space procedures of implicit regularization, the major principle behind this equivalence being the extension of the properties of regular distributions to regularized ones.

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1 Introduction

The problem of a simple regularization technique from the calculational point of view that respects gauge invariance and supersymmetry is still of great relevance, especially beyond one-loop level. The most simple and pragmatical regularization scheme known is dimensional regularization. It is gauge invariant, since the manifest gauge symmetry is not spoiled by the dimensional modification of the amplitude. This is so because in this dimensional modification, all the properties of regular integrals are retained, like the vanishing of the surface terms and the preservation of the vector algebra (see Sect. 3 of [1]). Nevertheless, this is not the case when the theory to be treated is supersymmetric. The dimensional modification spoils the symmetry between fermions and bosons. Dimensional reduction [2, 3] appeared as a new supersymmetric invariant version of this method. It only modifies the dimension of the integral and preserves the fields and the other mathematical objects in the proper dimension of the theory. Some important steps towards a rigorous and model independent generalization of dimensional reduction beyond one-loop order have been given [4], but all-order statements have not been established.

Differential renormalization (DR) [5] is a method that works in the proper dimension of the theory in coordinate space. It has been proved to be quite simple and powerful in various applications [6–20]. The original DR consists in the manipulation of singular distributions attributing to them properties of the regular ones. They are expressed in terms of a simpler singular function and then one performs its substitution by a renormalized one. In this procedure originates an arbitrary mass parameter for each different expression. When symmetries are involved, relations between these parameters are established in order to obtain a symmetric result. The constrained (version of) differential renormalization (CDR) [21] was developed in order to automatically satisfy the symmetries without the need of such adjustments at the end of the calculations. For this, a set of rules was stated, which are actually extensions of some additional properties of regular distributions to the singular ones. A series of applications of this technique was successfully carried out, which includes abelian and non-abelian gauge symmetry, supersymmetric theories and supergravity calculations [22–27].

Implicit regularization (IR) [28–30] is a momentum-space regularization method defined in the physical dimension of the underlying theory. The basic idea behind the method is, after implicitly assuming some (unspecified) regulating function as part of the integrand of divergent amplitudes, to extend all the properties of regular integrals to the regularized ones. An algebraic identity is used to expand the integrand and separate their regularization dependent parts from the finite one. Symmetries of the

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model, renormalization or phenomenological requirements determine arbitrary parameters introduced by this procedure. In fact, there is a special choice of parameters that automatically preserves the symmetries in all anomaly free cases we have studied [28–43]. The possibility of these parameters being fixed at the beginning of the calculation is desirable, since it considerably simplifies the application of the method. This results in a constrained (version of) implicit regularization (CIR).

The technique has been shown to be tailored to treat theories with parity violating objects in integer dimensions. This is the case of chiral and topological field theories. The ABJ anomaly [44, 45], and the radiative generation of a Chern–Simons-like term, which violates Lorentz and CPT symmetries [31, 32] are examples successful application of the technique. Moreover the method was shown to respect gauge invariance in both abelian and non-abelian theories at one-loop order [31–33, 36]. The calculation of the β-function of the massless Wess–Zumino model (at three loops) was also performed as a test of the procedure [35]. A non-trivial test in a supersymmetric model was performed, in which the anomalous magnetic moment of the lepton in supergravity was successfully calculated [39]. The extension of CIR to higher loop order has been implemented and has been applied in scalar [38] and gauge theories [43]. As for higher order calculations, differential renormalization, in its original form, has been used with success in scalar and gauge theories. CDR at one-loop order has been used as a guide in supersymmetric calculations at two-loop order [46, 47].

Constrained implicit regularization and constrained differential renormalization (CDR) are examples of regularization methods that work in the proper dimension of the theory. Both were shown to respect gauge invariance in abelian and non-abelian theories. The two techniques were also tested in non-trivial supersymmetric calculations yielding positive results. Besides, although they work in different spaces, the results are all identical. This fact suggests the possibility of equivalence between the two frameworks. In this paper, we show this equivalence by mapping the rules of CDR into the ones of CIR.

The paper is organized as follows: in Sect. 2, we present the basics of constrained differential renormalization; in Sect. 3, the basics of CIR is considered; the connections between the rules of the two techniques are analyzed in Sect. 4 and, finally, concluding comments are presented in Sect. 5.

2 Constrained differential renormalization

We reproduce here the basics of constrained differential renormalization (CDR). Given an amplitude in position space, it is written as a linear combination of derivatives of basic functions. The basic functions are products of scalar Feynman propagators with a differential operator acting in the last one. For example, the bubble and the triangular basic functions have, respectively, the general form

$$B_{m_1 m_2}[\mathcal{O}] = \Delta_{m_1}(x)\mathcal{O}_x\Delta_{m_2}(x) \qquad (1)$$

and

$$T_{m_1 m_2 m_3}[\mathcal{O}] = \Delta_{m_1}(x)\Delta_{m_2}(y)\mathcal{O}_x\Delta_{m_3}(x-y)\,, \qquad (2)$$

where $\Delta_m(x)$ is the scalar Feynman propagator and $\mathcal{O}_x$ is a differential operator with respect to x.

A crucial step in order to write the amplitude in this way is the use of the Leibniz rule for derivatives. The rules that we will list below permit one to write renormalized expressions for the basic functions such that, when they are substituted into the amplitude, we will have the underlying symmetries of the theory preserved.

The rules are as follows.

1. Differential reduction: singular expressions are substituted by derivatives of regular ones. For this, two steps are used.
   - Functions with singular behavior worse than logarithmic are reduced to derivatives of logarithmically singular functions without introducing any dimensionful constant.
   - For the logarithmically singular functions (at one loop) the following identity is used:

   $$\frac{1}{x^4} = -\frac{1}{4}\Box\frac{\ln x^2M^2}{x^2} \equiv \left(\frac{1}{x^4}\right)^{\mathrm{R}}\,. \qquad (3)$$

   This relation introduces the unique mass scale of the whole process. It plays the role of a renormalization group scale. The superscript R will make sense in connection with the next rule and it indicates that we are dealing with the renormalized basic function.
2. Formal integration by parts: derivatives act formally by parts on test functions. For a general basic function

   $$F[\mathcal{O}](x_1,\ldots,x_n) \equiv \Delta_{m_1}(x_1)\ldots\Delta_{m_n}(x_n) \times \mathcal{O}_{x_1}\Delta_{m_{n+1}}(x_1+x_2+\ldots+x_n)\,, \qquad (4)$$

   with $\mathcal{O}_{x_1}$ a differential operator with respect to x_1, we have

   $$[\partial F]^{\mathrm{R}} = \partial F^{\mathrm{R}}. \qquad (5)$$

   In words, this rule states that, when calculating the Fourier transform of a basic function and integration by parts is carried out, the surface term is discarded. This means validity of the equation above. So the superscript R makes sense, since the first rule states that the renormalized expression for a singular basic function is written in terms of derivatives of a expression with well defined Fourier transform.
3. Delta function renormalization rule: for the general basic function of (4), it is assumed that

   $$[F[\mathcal{O}](x,x_1,\ldots,x_n)\delta(x-y)]^{\mathrm{R}} = [F[\mathcal{O}](x,x_1,\ldots,x_n)]^{\mathrm{R}}\delta(x-y)\,. \qquad (6)$$

4. The validity of the propagator equation:

   $$\left[F[\mathcal{O}](x,x_1,\ldots,x_n)(\Box^x-m^2)\Delta_m(x)\right]^{\mathrm{R}} = [F[\mathcal{O}](x,x_1,\ldots,x_n)(-\delta(x)]^{\mathrm{R}}\,. \qquad (7)$$

With the rules above, one can find relations between the basic functions. A table with the renormalized basic functions can always be used to perform the calculation of any amplitude.

3 Constrained implicit regularization

Implicit regularization (IR) can be formulated by a similar set of rules, just like CDR. The first thing to be done is writing the momentum-space amplitude as a linear combination of basic integrals, multiplied by polynomials of the external momentum. These basic integrals are the Fourier transforms of the CDR basic functions. Typical basic integrals are

$$I, I_\mu, I_{\mu\nu} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1, k_\mu, k_\mu k_\nu}{(k^2-m^2)[(p-k)^2-m^2]}\,. \qquad (8)$$

They are, respectively, the Fourier transforms of $B_{mm}[1]$, $B_{mm}[\partial_\mu]$ and $B_{mm}[\partial_\mu \partial_\nu]$. As in the case of CDR, each one of these basic integrals can be treated following a set of rules. So a table with their results can be used whenever a new calculation is being performed. The rules of CIR are as follows.

1. A regularization technique is applied to the integral. It can be maintained implicit, but it must have some properties: it cannot modify the integrand and the dimension of the space-time. The first property is to preserve the finite part and the second one is a requirement in order not to violate supersymmetry. A good one would be a simple cutoff. The problem of possible violation of symmetries by this technique will be automatically handled by the constraining character of implicit regularization.
2. The divergent part to be subtracted in a given basic integral is obtained by applying recursively the identity

$$\frac{1}{(p-k)^2-m^2} = \frac{1}{(k^2-m^2)} - \frac{p^2-2p\cdot k}{(k^2-m^2)[(p-k)^2-m^2]}\,, \qquad (9)$$

until the divergent part does not have the external momentum p in the denominator. This will assure local counterterms. The remaining divergent integrals have the general form

$$\int_k^\Lambda \frac{k_{\mu_1} k_{\mu_2} \ldots}{(k^2-m^2)^\alpha}\,, \qquad (10)$$

where $\int_k$ stands for $\int \mathrm{d}^4 k/(2\pi)^4$ and the superscript Λ is to indicate that the integral is regularized. One more comment is in order. The assumption that a regularization is working in the basic integrals is, in fact, only for the separation of the finite part from the regularization dependent one (which does not need to be calculated), by means of the recursive application of the identity (9). This is analogous to the application of the Taylor operator on the integrand of an amplitude in the BPHZL method.
3. The divergent integrals with Lorentz indices must be expre ssed in function of surface terms. For example,

$$\int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)^3} = \frac{1}{4}\left(-\int_k^\Lambda \frac{\partial}{\partial k^\nu}\left(\frac{k_\mu}{(k^2-m^2)^2}\right) + g_{\mu\nu}\int_k^\Lambda \frac{1}{(k^2-m^2)^2}\right). \qquad (11)$$

The surface terms, which vanish for integrable cases, depend here on the regularization applied. They are symmetry violating terms. The possibility of making shifts in the integrals needs the surface terms to vanish. As far as loop integrals are concerned, non-null surface terms imply that the amplitude depends on the momentum routing choice. So the constraint of IR is the restoring of symmetry by means of the cancellation of these surface terms with local restoring counterterms. In practice, we do this automatically by setting them to zero. We will comment on the anomalous situation later.
4. The divergent part of the integral is written in terms of the basic divergences, thus:

$$I_{\log}(m^2) = \int_k^\Lambda \frac{1}{(k^2-m^2)^2} \qquad (12)$$

and

$$I_{\mathrm{quad}}(m^2) = \int_k^\Lambda \frac{1}{(k^2-m^2)}\,. \qquad (13)$$

These objects will require local counterterms in the process of renormalization.

Finally, we can solve the finite (regularization independent) part and define a subtraction scheme, for instance, absorbing the basic divergent integrals in the renormalization constants defined by the counterterms. This can be done in a mass independent fashion. For this, we use a scale relation between the basic divergent integrals, which will also introduce the renormalization group scale of the method.

In order to give an example of the use of these steps, we apply the method to the simple logarithmically divergent one-loop amplitude

$$I = \int^\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2-m^2)[(k-p)^2-m^2]}. \qquad (14)$$

By applying the identity (9) in the regularized amplitude above, we get

$$I = I_{\log}(m^2) - \int_k \frac{p^2-2p\cdot k}{(k^2-m^2)^2[(k-p)^2-m^2]}\,. \qquad (15)$$

Notice that the second integral in (15) is finite and, because of this, we do not use the superscript Λ. It is convenient to express the regularization dependent part, given by (12), in terms of an arbitrary mass parameter, λ. This becomes essential if we are treating massless theories [38]. It can be done by using the regularization independent relation

$$I_{\log}(m^2) = I_{\log}(\lambda^2) + b \ln\left(\frac{m^2}{\lambda^2}\right), \tag{16}$$

with $b = \mathrm{i}/(4\pi)^2$. The mass parameter λ^2 is suitable for use as the renormalization group scale, as can be seen in [35, 38]. After solving the finite part, we are left with

$$I = I_{\log}(\lambda^2) - bZ_0(p^2, m^2, \lambda^2), \tag{17}$$

where

$$Z_0(p^2, m^2, \lambda^2) = \int_0^1 \mathrm{d}x \ln\left(\frac{p^2 x(1-x) - m^2}{(-\lambda^2)}\right). \tag{18}$$

Finally, we would like to comment on the relation between surface terms and anomalies. Momentum routing invariance seems to be the crucial property in a Feynman diagram in order to preserve symmetries. In fact such surface terms evaluate to zero should we employ dimensional regularization (DREG) to explicitly evaluate them. This property somewhat reveals why DREG is manifestly gauge invariant; yet it breaks supersymmetry (the invariance of the action with respect to supersymmetry transformations only holds in general for specific values of the space-time dimension).[1]

A particular situation, however, is the occurrence of quantum symmetry breaking (anomaly). Anomalies, within perturbation theory, may present some oddities such as preserving a certain symmetry at the expense of adopting a special momentum routing in a Feynman diagram e.g. in the (Adler–Bardeen–Bell–Jackiw) AVV triangle anomaly. In the case of chiral anomalies, IR has been shown to preserve the democracy between the vector and axial sectors of the Ward identities, which is a good 'acid test' for regularizations [32]. The arbitrary parameter represented by the surface term remains undetermined and floats between the axial and vector sectors of the Ward identities. That is to say, in the anomalous amplitudes, there is no possibility of restoring, at the same time, the axial and the vectorial Ward identities. The counterterm that will restore one symmetry causes the violation of the other and, therefore, it does not make sense to set the surface terms to zero. The answer is to be established by physical constraints on such an amplitude. This feature has also been illustrated in the description of two-dimensional gravitational anomalies [37].

[1] The idea of associating momentum routing in the loops with symmetry properties of the Green's functions has been exploited in a framework named 'preregularization', which did not call for momentum routing invariance but instead fixed the routing in order to fulfill certain Ward identities [48].

4 Mapping constrained implicit regularization in constrained differential renormalization

We show in this section that the rules of constraining differential renormalization can be mapped in the ones of constrained implicit regularization. We will sometimes reproduce with few details the calculations of [25].

Rules 1 and 2 of CDR

We begin by analyzing rules 1 and 2 of CDR. We will consider here the simpler two-point massless basic function, $B[1]$. The reason is that the one-loop renormalization of CDR will always occur when the other basic functions are written as functions of it. So its renormalization is the basis for finding all the other renormalized expressions. We write

$$B[1] = \Delta(x) 1 \Delta(x) = \left(\frac{1}{4\pi^2 x^2}\right)^2, \tag{19}$$

which, after application of rule 1, gives

$$B^R[1] = -\frac{1}{4}\left(\frac{1}{4\pi^2}\right)^2 \Box \frac{\ln x^2 M^2}{x^2}. \tag{20}$$

In order to compare the two techniques, we will take this basic function into the momentum space. The bare momentum-space expression for $B[1]$ in euclidian space is given by

$$\hat{B}[1] = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 (p-k)^2} = I_{\mathrm{E}}, \tag{21}$$

where $I_{\mathrm{E}} = -\mathrm{i}I$ of (14) for the massless case. If we intend to obtain the Fourier transform of the renormalized expression, we will have to make use of CDR rule 2. This rule says that we must ignore the surface term when integration by parts is performed. With this prescription, the derivatives act directly on the exponential. So we get

$$\hat{B}^{\mathrm{R}}[1] = -\frac{1}{(4\pi)^2} \ln\left(\frac{p^2}{\bar{M}^2}\right), \tag{22}$$

with $\bar{M}^2 = 4M^2/\gamma^2$, γ being the Euler constant.

We would like here to show that rules 1 and 2 together stands for the subtraction of $I_{\log}(\lambda^2)$ in implicit regularization. To make it clear, we reproduce here the calculation of [5], in which the authors show that the combination of rules 1 and 2 corresponds to the subtraction of a local cut-off dependent term. Let us consider the exclusion of a small ball, $\mathcal{B}_\epsilon$, of radius ϵ. We can write

$$\begin{aligned}
\hat{B}[1] &= \int_{R^4 - \mathcal{B}_\epsilon} \mathrm{d}^4 x f(x) \left(\frac{1}{4\pi^2 x^2}\right)^2 \\
&= -\frac{1}{4}\left(\frac{1}{4\pi^2}\right)^2 \int_{R^4 - \mathcal{B}_\epsilon} \mathrm{d}^4 x f(x) \Box \frac{\ln x^2 M^2}{x^2} \\
&= \frac{1}{4}\left(\frac{1}{4\pi^2}\right)^2 \left\{ \int_{S_\epsilon} \mathrm{d}\sigma_\mu \ f(x) \partial_\mu \frac{\ln x^2 M^2}{x^2} \right. \\
&\qquad \left. + \int_{R^4 - \mathcal{B}_\epsilon} \mathrm{d}^4 x \partial_\mu f(x) \partial_\mu \frac{\ln x^2 M^2}{x^2} \right\},
\end{aligned} \tag{23}$$

with $\mathrm{d}\sigma_\mu = \epsilon^3 \hat{x}_\mu \,\mathrm{d}\hat{x}$ the outward normal volume element of the 3-sphere S_ϵ, which is the boundary of $\mathcal{B}_\epsilon$. The second integral is well defined and can be integrated by parts with no problem. The complete result, taking in account that $f(x) = \mathrm{e}^{\mathrm{i}p\cdot x}$, is

$$\hat{B}[1] = \frac{1}{(4\pi)^2}\left(1 - \ln \epsilon^2 M^2 - \ln\left(\frac{p^2}{\bar{M}^2}\right)\right) = \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\Lambda^2}{\bar{M}^2}\right) + 1 - \ln\left(\frac{p^2}{\bar{M}^2}\right)\right), \quad (24)$$

where the momentum cutoff is given by $\Lambda^2 = 4/(\gamma^2\epsilon^2)$.

We now remember that $\hat{B}[1] = I_\mathrm{E}$ and use the implicit regularization result of (14) for $m^2 = 0$, so that

$$\hat{B}[1] = -\mathrm{i}\left\{I_{\log}(\lambda^2) - b \ln\left(\frac{p^2}{\mathrm{e}^2\lambda^2}\right)\right\}. \quad (25)$$

If we use a simple momentum cutoff, Λ^2, to calculate $I_{\log}(\lambda^2)$, we get

$$\hat{B}[1] = \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\Lambda^2}{\mathrm{e}^2\lambda^2}\right) + 1 - \ln\left(\frac{p^2}{\mathrm{e}^2\lambda^2}\right)\right). \quad (26)$$

It is the same as the result of differential renormalization. We just have to rescale our mass parameter such that $\mathrm{e}^2\lambda^2 = \bar{M}^2$. The important conclusion here is that the position-space surface term that is subtracted by means of rule 2 of CDR is exactly the basic divergence of IR, $I_{\log}(\lambda^2)$.

In (2.15) of [5], the authors find a divergent contribution in the limit $\epsilon \to 0$, associated to the radius ϵ of a spherical surface around the propagator distributional product pole. Such a divergent counterpart could be subtracted by adding a suitable counterterm in an arbitrary proportion to the action. An alternative procedure would be to consider the pole contribution as a concentrated distribution at $x = 0$. There arises a linear combination of delta functions with arbitrary coefficients [49], as many as the pole order is higher. In both cases the arbitrariness is fixed by a subtraction scheme.

The role of rules 3 and 4 of CDR

Next, we dedicate this subsection to the role of rules 3 and 4 in constrained differential renormalization in order to understand how they are translated to implicit regularization. These rules are important for introducing an unique mass parameter in the calculation that is being performed. They make it possible to establish relations between a basic function with $n+1$ propagators with one with n propagators. If this one is already renormalized, then the same mass scale is used. Let us see how it works before we look at its version in IR. Consider the basic function

$$F_n[\Box - m_{n+1}^2] = \Delta_{m_1}(x_1)\ldots\Delta_{m_n}(x_n) \times \left(\Box^{x_1} - m_{n+1}^2\right) \times \Delta_{m_{n+1}}(x_1 + x_2 + \ldots + x_n), \quad (27)$$

where $F_n[\mathcal{O}] \equiv F[\mathcal{O}](x_1, x_2, \ldots, x_n)$ and the variables $x_1, \ldots, x_n$ are differences between the vertex points of the loop. With the help of rule 4, we can write

$$F_n^{\mathrm{R}}\left[\Box - m_{n+1}^2\right] = -(F_{n-1}[1]\delta(x_1 + x_2 + \ldots + x_n))^{\mathrm{R}} \quad (28)$$

and, by using rule 3:

$$F_n^{\mathrm{R}}\left[\Box - m_{n+1}^2\right] = -F_{n-1}^{\mathrm{R}}[1]\delta(x_1 + x_2 + \ldots + x_n). \quad (29)$$

In momentum space, a momentum k_i is associated with each internal line and therefore with each variable x_i. If we consider as the loop momentum the one associated with the last propagator, we have

$$\hat{F}_{n-1}^{\mathrm{R}}[\mathcal{O}_k](p_1, \ldots, p_{n-1}) = \left(\int_k \frac{\mathcal{O}_k}{(k^2 + m_n^2)[(k - p_1)^2 + m_1^2]\ldots[(k - p_{n-1})^2 + m_{n-1}^2]}\right)^{\mathrm{R}}. \quad (30)$$

Nevertheless, Fourier transforming (29) yields

$$\left(\int_k \frac{(k^2 + m_{n+1}^2)}{(k^2 + m_{n+1}^2)\left[(k - p_1)^2 + m_1\right]^2 \ldots [(k - p_n)^2 + m_n^2]}\right)^{\mathrm{R}} = \left(\int_k \frac{1}{(k^2 + m_n^2)\left[(k + p_n - p_1)^2 + m_1^2\right]}\cdots \times \frac{1}{[(k + p_n - p_{n-1})^2 + m_{n-1}^2]}\right)^{\mathrm{R}}. \quad (31)$$

The straightforward operation in momentum space, which is standard in CIR, is the simple cancellation of the factor $(k^2 + m_{n+1}^2)$, present in both numerator and denominator of the integrand. But the sequence of procedures performed above includes a shift $k \to k + p_n$ in the integral. It is also in accordance with CIR, where surface terms are discarded. Let us see how rules 3 and 4 perform this shift. We can write (28), in which rule 3 has not yet been applied, in terms of its inverse Fourier transform, as

$$\begin{aligned} F_n^{\mathrm{R}}\left[\Box - m_{n+1}^2\right] &= -\left(\int_{p_1,\ldots,p_{n-1}} \hat{F}_{n-1}[1](p_1, \ldots, p_{n-1}) \times \mathrm{e}^{\mathrm{i}p_1\cdot x_1}\ldots\mathrm{e}^{\mathrm{i}p_{n-1}\cdot x_{n-1}}\delta\left(\sum x_i\right)\right)^{\mathrm{R}} \\ &= -\left(\int_{p_1,\ldots,p_{n-1}}\int_k \frac{1}{(k^2 + m_n^2)[(k - p_1)^2 + m_1^2]}\cdots \times \frac{1}{[(k - p_{n-1})^2 + m_{n-1}^2]}\mathrm{e}^{\mathrm{i}p_1\cdot x_1}\ldots \times \mathrm{e}^{\mathrm{i}p_{n-1}\cdot x_{n-1}}\delta\left(\sum x_i\right)\right)^{\mathrm{R}}. \end{aligned} \quad (32)$$

Carrying out the Fourier transformation of the equation above and applying rule 3, we get

$$\begin{aligned}\hat{F}_n^{\mathrm{R}}\left[-(k^2+m_{n+1}^2)\right] &= -\int_{x_1,\dots,x_n}\left(\int_{p_1,\dots,p_{n-1}}\int_k \frac{1}{(k^2+m_n^2)[(k-p_1)^2+m_1^2]}\right.\\ &\quad\times\dots\frac{1}{[(k-p_{n-1})^2+m_{n-1}^2]}\mathrm{e}^{\mathrm{i}(p_1-p_1')\cdot x_1}\dots\\ &\quad\left.\times\,\mathrm{e}^{\mathrm{i}(p_{n-1}-p_{n-1}')\cdot x_{n-1}}\mathrm{e}^{-\mathrm{i}p_n'\cdot x_n}\right)^{\mathrm{R}}\delta\left(\sum x_i\right). \qquad (33)\end{aligned}$$

Integration on the x variables gives us

$$\begin{aligned}\hat{F}_n^{\mathrm{R}}\left[-(k^2+m_{n+1}^2)\right] &= -\left(\int_{p_1,\dots,p_{n-1}}\int_k \frac{1}{(k^2+m_n^2)[(k-p_1)^2+m_1^2]}\dots\right.\\ &\quad\times\frac{1}{[(k-p_{n-1})^2+m_{n-1}^2]}\\ &\quad\left.\times\,\delta(p_1-p_1'+p_n')\dots\delta(p_{n-1}-p_{n-1}'+p_n')\right)^{\mathrm{R}}. \qquad (34)\end{aligned}$$

This will furnish us with the same as the result of (31).

It is clear from the expression above that rules 3 and 4 taken together are equivalent, in momentum space, to the cancellation of a factor $k^2+m_{n+1}^2$ (here, in euclidian space) in the numerator with its correspondent in the denominator and the subsequent shift $k \to k+p_n$. A comment is in order. There is physical appeal in the result above. The operation we are discussing corresponds to a point contraction. If we consider the original outgoing external momenta $p_1,\dots,p_n$, we have $\sum p_i = 0$. When the point contraction is performed, the momentum p_n does not flow outward, so that the internal momentum that circulates the loop is changed to $k+p_n$. Alternatively, if we consider the definition (30), we have for the result of (31): $-\hat{F}_{n-1}^{\mathrm{R}}[1](p_1-p_n,\dots,p_{n-1}-p_n)$. This is in accordance with the new condition of energy-momentum conservation, $p_1+\dots+p_{n-1}=0$.

We have seen in this calculation that the sequence of applications of rules 3 and 4, when observed from the momentum space, includes a shift. Clearly, if the integral is at least linearly divergent, this corresponds to discarding a surface term. But, as we will show below, this is not the unique procedure of CDR that works as a source of shifts in momentum space. Besides, generally (but not always) rules 3 and 4 are used with the intention of using the renormalized version of the basic function $B[1]$, which corresponds to a logarithmically divergent integral in momentum space. For this case, no surface term is missed. As we shall see, the crucial point occurs when Lorentz indices are involved and Leibniz rule is used.

Finally, we enforce that, in constrained implicit regularization, shifts and the cancellation of factors of the numerator and the denominator are essential steps. With these procedures, we can always display the basic divergences as the $I_{\log}$ and the I_{quad}, which depend on the same mass parameter.

Leibniz rule in position space, ambiguities in Fourier transforms and shifts in momentum space

There is an essential characteristic of constrained differential renormalization, which we will show, that takes care of momentum-space surface terms: the validity of the Leibniz rule. It is an essential tool when one establishes relations between basic functions with and without Lorentz indices. This happens in connection with an ambiguity when the Fourier transform of a bare basic function is performed. Let us consider the basic function,

$$\begin{aligned}F[\partial_\mu](x_1,x_2,\dots,x_n) &= \Delta_{m_1}(x_1)\dots\Delta_{m_n}(x_n)\\ &\quad\times\partial_\mu^{x_1}\Delta_{m_{n+1}}(x_1+x_2+\dots x_n),\end{aligned} \qquad (35)$$

which has the Fourier transform

$$\begin{aligned}\hat{F}[k_\mu](p_1,\dots,p_n) &= \int_{k_1,\dots,k_{n+1}}\frac{ik_\mu^{n+1}}{(k_1^2+m_1^2)\dots(k_{n+1}^2+m_{n+1}^2)}\\ &\quad\times\delta(k_1+k_{n+1}-p_1)\dots\delta(k_n+k_{n+1}-p_n). \qquad (36)\end{aligned}$$

At this point, if there is a singularity, there emerges an ambiguity: depending on the momentum we choose to be the loop momentum, a different momentum routing is obtained. In other words, the integrals will differ by a shift. First, let us take k_{n+1} to be the loop momentum. We obtain

$$\begin{aligned}\hat{F}[k_\mu](p_1,\dots,p_n) &= \int_k \frac{\mathrm{i}k_\mu}{(k^2+m_{n+1}^2)\,[(k-p_1)^2+m_1^2]\dots[(k-p_n)^2+m_n^2]}\\ &= \int_k \frac{-\mathrm{i}k_\mu}{(k^2+m_{n+1}^2)\,[(k+p_1)^2+m_1^2]\dots[(k+p_n)^2+m_n^2]}.\end{aligned} \qquad (37)$$

The last equality follows from the Lorentz structure of the integral. On the other hand, if we choose k_1, we have

$$\hat{F}[k_\mu](p_1,\dots,p_n) = \int_k \frac{\mathrm{i}(p_1-k)_\mu}{(k^2+m_1^2)\,[(k-p_1)^2+m_{n+1}^2]\dots[(k+p_n-p_1)^2+m_n^2]}. \qquad (38)$$

It is clear that (38) is obtained by performing the shift $k \to k-p_1$ in the integrand of (37). There is nothing wrong with this if the integral is finite or at most logarithmically divergent. But this is not the case in general. If the integral is linearly divergent, for instance, a surface term must be added to compensate for the shift. One could avoid this problem by stating, as a rule of the technique, that the momentum associated with the last propagator, which closes the loop, should be the loop momentum.

Nevertheless, in some situations, if the Leibniz rule is allowed for, this ambiguity cannot be removed. Let us consider the simple example of the massless basic function,

$$B[\partial_\mu] = \Delta(x)\partial_\mu\Delta(x)\,, \tag{39}$$

which, by the Leibniz rule, can be written as

$$B[\partial_\mu] = \frac{1}{2}\partial_\mu B[1]\,, \tag{40}$$

so that

$$B^{\mathrm{R}}[\partial_\mu] = \frac{1}{2}\partial_\mu B^{\mathrm{R}}[1]\,. \tag{41}$$

We should call the reader's attention to the fact that (40) was written considering that

$$\Delta(x)\partial_\mu\Delta(x) = \partial_\mu(\Delta(x))\Delta(x)\,, \tag{42}$$

which, if we take into account the rule discussed above, which tells us that the momentum associated to the last propagator is the loop momentum, implies that

$$-\int_k \frac{\mathrm{i}k_\mu}{k^2(p+k)^2} = \int_k \frac{\mathrm{i}(p-k)_\mu}{k^2(p-k)^2}\,. \tag{43}$$

So we can say that the application of the Leibniz rule in position space, in some peculiar situations, is equivalent to discarding a surface term in momentum space.

In the discussion that follows in this section, we will show that all the relations between the basic functions that in CDR are obtained by using Leibniz and its rules 3 and 4 can be obtained in momentum space by performing shifts and by canceling common factors of the numerator and the denominator. We return to (41). If we look at this equation in momentum space, we have

$$\left(\int_k \frac{k_\mu}{k^2(p-k)^2}\right)^{\mathrm{R}} = \frac{1}{2}p_\mu\left(\int_k \frac{1}{k^2(p-k)^2}\right)^{\mathrm{R}}, \tag{44}$$

or $I_\mu = (1/2)p_\mu I$. The relation above was obtained by the use of properties of regular distributions in position space extended to singular ones. We would like to treat directly in momentum space the integral I_μ. We will make use of two different ways of calculation. In the first one, considering the extension of all the properties of regular to regularized integrals, we perform the shift $k \to k+p$ in the integral I_μ, so that

$$I_\mu = \int_k \frac{(k+p)_\mu}{k^2(p+k)^2} = \int_k \frac{k_\mu}{k^2(p+k)^2} + p_\mu I = -I_\mu + p_\mu I\,. \tag{45}$$

In the last step, we have observed that I_μ is odd in p. The equation above leads us again to the result $I_\mu = (1/2)p_\mu I$. In the procedure above we have shifted a linear divergent integral. This would require the addition of a surface term. So this step is in accordance with the rule of implicit regularization that tells us to eliminate such terms by means of symmetry restoring counterterms.

The second way we treat I_μ is its explicit calculation by means of implicit regularization. This will permit us to identify the forgotten surface term. We begin applying the identity (9) twice to the integrand:

$$\begin{aligned} I_\mu = \int_k^\Lambda \frac{k_\mu}{(k^2-m^2)}\left(\frac{1}{(k^2-m^2)} - \frac{p^2-2p\cdot k}{(k^2-m^2)^2}\right. \\ \left. + \frac{(p^2-2p\cdot k)^2}{(k^2-m^2)^2[(p-k)^2-m^2]}\right)\,, \end{aligned} \tag{46}$$

in which we will take the limit $m^2 \to 0$ at the end of the calculation. Eliminating the vanishing terms, we have

$$\begin{aligned} I_\mu &= 2p^\alpha \int_k^\Lambda \frac{k_\mu k_\alpha}{(k^2-m^2)^3} + \int_k \frac{k_\mu(p^2-2p\cdot k)^2}{(k^2-m^2)^3[(p-k)^2-m^2]} \\ &= \frac{p^\alpha}{2}\left(-\int_k^\Lambda \frac{\partial}{\partial k^\mu}\left(\frac{k_\alpha}{(k^2-m^2)^2}\right) + g_{\mu\alpha}\int_k^\Lambda \frac{1}{(k^2-m^2)^2}\right) \\ &\quad + \tilde{I}_\mu\,, \end{aligned} \tag{47}$$

$\tilde{I}_\mu$ being the finite integral. After the calculation of this finite part, we obtain, in the limit $m^2 \to 0$,

$$\begin{aligned} I_\mu &= \frac{p_\mu}{2}\left(I_{\log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2\mathrm{e}^2}\right)\right) - \frac{p^\alpha}{2}S_{\mu\alpha} \\ &= \frac{p_\mu}{2}I - \frac{p^\alpha}{2}S_{\mu\alpha}\,, \end{aligned} \tag{48}$$

where

$$S_{\mu\alpha} = \int_k^\Lambda \frac{\partial}{\partial k^\mu}\left(\frac{k_\alpha}{(k^2-m^2)^2}\right) \tag{49}$$

is a surface term that will be set to zero and where we have made use of the scale relation (16). In the analysis above, we saw that the validity of the Leibniz rule in the calculation of $B[\partial_\mu]$ implies the validity of a shift in a linear divergent integral. In other words, it means that a surface term has been subtracted. Although we are analyzing a particular case, this relation is used in the derivation of all basic functions with upper Lorentz indices, as we show in some examples.

The next example we examine is the calculation of the basic function,

$$T[\partial_\mu\partial_\nu] = \Delta(x)\Delta(y)\partial^x_\mu\partial^x_\nu\Delta(x+y)\,. \tag{50}$$

It can be decomposed into a traceless and a trace part. A local term is added and is to be fixed, due to a possible ambiguity in the finite traceless part:

$$\begin{aligned} T^{\mathrm{R}}[\partial_\mu\partial_\nu] = T\left[\partial_\mu\partial_\nu - \frac{1}{4}\delta_{\mu\nu}\Box\right] + \frac{1}{4}\delta_{\mu\nu}T^{\mathrm{R}}[\Box] \\ + \frac{1}{64\pi^2}c\delta_{\mu\nu}\delta(x)\delta(y)\,. \end{aligned} \tag{51}$$

In the equation above the second term of the r.h.s. is renormalized by means of rules 3 and 4 of CDR and c is the arbitrary constant to be fixed. It is fixed so that the rules of CDR are valid. Specifically the Leibniz rule and rules 3 and 4 of CDR, state that

$$B^{\mathrm{R}}[\partial_\mu](x)\delta(y) = -\Box^y T[\partial_\mu] + 2\partial_\rho^y T^{\mathrm{R}}[\partial_\mu\partial_\rho] - T^{\mathrm{R}}[\Box\partial_\mu]\,. \tag{52}$$

The integration of this equation over x yields $c = -1/2$. We will repeat this procedure in momentum space and show that the constant c is fixed so that it cancels the surface term that comes from the traceless part. We have

$$\hat{T}^{\mathrm{R}}[k_\mu k_\nu] = \hat{T}\left[k_\mu k_\nu - \frac{1}{4}g_{\mu\nu}k^2\right] + \frac{1}{4}g_{\mu\nu}\hat{T}^{\mathrm{R}}[k^2] + \frac{1}{64\pi^2}cg_{\mu\nu}\,. \tag{53}$$

We remember that an infrared cutoff m^2 is used (it disappears after the scale relation is used). By using (9) and the identity $k^2 = (k^2 - m^2) + m^2$ in the first term, we get

$$-\frac{1}{4}\left\{\int_k^\Lambda \frac{g_{\mu\nu}}{(k^2-m^2)^2} - 4\int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)^3}\right\} + \frac{1}{64\pi^2}cg_{\mu\nu} + \text{non-ambiguous terms}\,. \tag{54}$$

We calculate it by symmetric integration ($k_\mu k_\nu \to k^2 g_{\mu\nu}/4$):

$$\begin{aligned} -\frac{1}{4}&\left\{\int_k^\Lambda \frac{g_{\mu\nu}}{(k^2-m^2)^2} - 4\int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)^3}\right\} + \frac{1}{64\pi^2}cg_{\mu\nu} \\ &= \frac{m^2}{4}g_{\mu\nu}\int_k \frac{1}{(k^2-m^2)^3} + \frac{1}{64\pi^2}cg_{\mu\nu} \\ &= -\frac{\mathrm{i}g_{\mu\nu}}{128\pi^2} + \frac{1}{64\pi^2}cg_{\mu\nu}\,. \end{aligned} \tag{55}$$

So in order to cancel this surface term, $c = \mathrm{i}/2$, just like the result of constrained differential renormalization (the i factor is due to the fact that we work in Minkowski space).

The two examples we have worked out above are cases that involve at most linear divergences. Besides, the number of Lorentz indices is at most two. When the degree of divergence or the number of Lorentz indices increases, new surface terms appear. Let us see the case of the basic function,

$$B[\partial_\mu\partial_\nu] = \Delta(x)\partial_\mu\partial_\nu\Delta(x)\,. \tag{56}$$

Following the steps of CDR, its most general renormalized expression is given by

$$B[\partial_\mu\partial_\nu] = \frac{1}{3}\left(\partial_\mu\partial_\nu - \frac{1}{4}\delta_{\mu\nu}\Box\right)B^{\mathrm{R}}[1] + \frac{1}{16\pi^2}\left[f\partial_\mu\partial_\nu + \delta_{\mu\nu}(g\Box + \mu^2)\right]\delta(x)\,, \tag{57}$$

where a differential equation was solved to find the first term at $x \neq 0$ and the constants f, g (dimensionless) and μ (mass dimension) where introduced to take care of ambiguities. As before, they will be fixed in such a way that the rules of CDR, including the Leibniz rule, are respected. Using these laws of manipulation, it can be shown that

$$-\frac{1}{2}\Box\partial_\mu B^{\mathrm{R}}[1] + 2\partial_\rho B^{\mathrm{R}}[\partial_\mu\partial_\rho] = 0 \tag{58}$$

and

$$\begin{aligned} B^{\mathrm{R}}[\partial_\mu\partial_\nu](x)\delta(y) = &-\Box^y T^{\mathrm{R}}[\partial_\mu\partial_\nu] + 2\partial_\rho T^{\mathrm{R}}[\partial_\mu\partial_\nu\partial_\rho] \\ &+ \frac{1}{2}\left(\partial_\mu^x\partial_\nu^y + \partial_\nu^x\partial_\mu^y\right)\left(B^{\mathrm{R}}[1](x)\delta(x+y)\right) \\ &+ R^{\mathrm{B}}[\partial_\mu\partial_\nu](x)\delta(x+y)\,. \end{aligned} \tag{59}$$

We will also need the basic function, which in the equation below is decomposed in a trace and a traceless part, plus a local arbitrary term, left to be adjusted according to the rules:

$$\begin{aligned} &T^{\mathrm{R}}[\partial_\mu\partial_\nu\partial_\rho] \\ &\quad= T[\partial_\mu\partial_\nu\partial_\rho - \frac{1}{6}(\delta_{\mu\nu}\partial_\rho + \delta_{\mu\rho}\partial_\nu + \delta_{\rho\nu}\partial_\mu)\Box] \\ &\qquad+ \frac{1}{12}\left(\delta_{\mu\nu}(\partial_\rho^x + \partial_\rho^y) + \delta_{\mu\rho}(\partial_\nu^x + \partial_\nu^y) + \delta_{\rho\nu}(\partial_\mu^x + \partial_\mu^y)\right) \\ &\qquad\times\left(-B^{\mathrm{R}}[1]\delta(x+y) + \frac{1}{16\pi^2}d\delta(x)\delta(y)\right)\,. \end{aligned} \tag{60}$$

We should notice that rules 3 and 4 and the renormalization of the basic function $B[\partial_\mu]$ have already been applied in the trace part. When (57) is substituted into (58), it is found that $\mu = 0$ and $g = -f$. If (57), (60) and (53) are inserted into the expression (59) and integration over x is carried out, the result is $f = \frac{1}{18}$ and $d = -\frac{1}{3}$.

Let us now see how it works in momentum space if the principles of constrained implicit regularization are considered. First, we show that (58) is respected in momentum space, as long as shifts in the integrand are permitted. In momentum space, we have

$$-\frac{1}{2}p^2 p_\mu I + 2\int_k^\Lambda \frac{(p\cdot k)k_\mu}{k^2(p-k)^2} = 0\,. \tag{61}$$

In the second integral, we can use

$$(p\cdot k) = -\frac{1}{2}[(p-k)^2 - k^2 - p^2]\,, \tag{62}$$

so that it is given by

$$-\left\{\int_k^\Lambda \frac{k_\mu}{k^2} - \int_k^\Lambda \frac{k_\mu}{(p-k)^2} - p^2 I_\mu\right\}\,. \tag{63}$$

The first term is obviously null. We shift the second one ($k \to k+p$) and obtain

$$p_\mu\int_k^\Lambda \frac{1}{k^2} + p^2 I_\mu = p_\mu I_{\mathrm{quad}}(m^2=0) + \frac{p^2}{2}p_\mu I\,. \tag{64}$$

In the expression above, the last term cancels out the first one of (61). For the basic quadratic divergence with null mass, in [32, 38] it was shown that an adequate parametrization gives us

$$I_{\text{quad}}(m^2) = \frac{\mathrm{i}}{(4\pi)^2} m^2 \left[\ln\left(\frac{\Lambda^2}{m^2}\right) + \text{const} \right], \tag{65}$$

so that $I_{\text{quad}}(m^2=0)=0$ and (61) is satisfied. We call the reader's attention to the fact that this parametrization furnishes the same result as the one of its correspondent in position space, the one-point basic function $A_m[1] = \Delta_m(x)\delta(x)$ of CDR. The next equation to be analyzed in momentum space is (59), given by

$$\begin{aligned}
-\int_k^\Lambda \frac{k_\mu k_\nu}{k^2(p-k)^2} &= -p'^2 \int_k^\Lambda \frac{k_\mu k_\nu}{k^2(p-k)^2(p'-k)^2} \\
&\quad + \int_k^\Lambda \frac{(2p'\cdot k) k_\mu k_\nu}{k^2(p-k)^2(p'-k)^2} \\
&\quad - \frac{1}{2}(p_\mu p'_\nu + p_\nu p'_\mu) \int_k^\Lambda \frac{1}{k^2(p-p'-k)^2} \\
&\quad - \int_k^\Lambda \frac{k_\mu k_\nu}{k^2(p-p'-k)^2}\,.
\end{aligned} \tag{66}$$

We begin by noting that the two first terms of the r.h.s. can be considered together, so that we have, in the numerator, $-p'^2 + 2(p'\cdot k) = k^2 - (p'-k)^2$. We break it again in two parts, and perform cancellations with factors of the denominator. So

$$\begin{aligned}
-\int_k^\Lambda \frac{k_\mu k_\nu}{k^2(p-k)^2} &= -\int_k^\Lambda \frac{k_\mu k_\nu}{k^2(p-k)^2} \\
&\quad + \int_k^\Lambda \frac{k_\mu k_\nu}{(p'-k)^2(p-k)^2} \\
&\quad - \frac{1}{2}(p_\mu p'_\nu + p_\nu p'_\mu) \int_k^\Lambda \frac{1}{k^2(p-p'-k)^2} \\
&\quad - \int_k^\Lambda \frac{k_\mu k_\nu}{k^2(p-p'-k)^2}\,.
\end{aligned} \tag{67}$$

The three last terms of the above equation must cancel out in order to satisfy the equation. If we perform the shift $k \to k-p'$ in the last two integrals and use the result, also obtained by means of shifts,

$$\int_k^\Lambda \frac{k_\mu}{(p'-k)^2(p-k)^2} = \frac{(p+p')_\mu}{2} \int_k^\Lambda \frac{1}{(p'-k)^2(p-k)^2}\,, \tag{68}$$

the exact cancellation occurs. The reader should observe that all the results that CDR reaches with the help of the Leibniz rule and its rules 3 and 4 are reached in momentum space with the use of shifts and cancellation of common factors of the numerator and the denominator.

We now will fix the arbitrary constants with the help of the principles of constrained implicit regularization. First, once (61) was verified, it is trivial to check that the momentum-space version of (57) implies $f=-g$ and $\mu = 0$. We now turn to the Fourier transform of the expression (60).

$$\begin{aligned}
&\hat{T}^{\mathrm{R}}[k_\mu k_\nu k_\rho] \\
&\quad = \hat{T}\left[k_\mu k_\nu k_\rho - \frac{1}{6}(g_{\mu\nu}k_\rho + g_{\mu\rho}k_\nu + g_{\rho\nu}k_\mu)k^2\right] \\
&\quad + \frac{1}{12}\left(g_{\mu\nu}\left(p_\rho + p'_\rho\right) + g_{\mu\rho}\left(p_\nu + p'_\nu\right) + g_{\rho\nu}\left(p_\mu + p'_\mu\right)\right) \\
&\quad \times \left(-\hat{B}^{\mathrm{R}}[1]((p-p')^2) + \frac{1}{16\pi^2}d\right).
\end{aligned} \tag{69}$$

As discussed before, the ambiguity concerns the traceless part, and it is due to the presence of surface terms. We recall that in (60) the trace part is already renormalized and, in this process, surface terms were discarded. Nevertheless, in the present form, there is no ambiguity in the trace part and the adjustment of the constant d is done considering this fact. So d must be adjusted so as to cancel the surface terms coming from the traceless part. We can write

$$\begin{aligned}
&\hat{T}^{\mathrm{R}}[k_\mu k_\nu k_\rho] \\
&= J_{\mu\nu\rho} - \frac{1}{6}\left(g_{\mu\nu}I_\rho(p,p') + g_{\mu\rho}I_\nu(p,p') + g_{\nu\rho}I_\mu(p,p')\right) \\
&\quad + \frac{1}{12}\left(g_{\mu\nu}(p_\rho + p'_\rho) + g_{\mu\rho}(p_\nu + p'_\nu) + g_{\rho\nu}(p_\mu + p'_\mu)\right)\frac{1}{16\pi^2}d \\
&\quad + \text{non-ambiguous terms}\,.
\end{aligned} \tag{70}$$

Let us consider the first integral,

$$\begin{aligned}
J_{\mu\nu\rho} &= \int_k^\Lambda \frac{k_\mu k_\nu k_\rho}{(k^2-m^2)[(p'-k)^2-m^2][(p-k)^2-m^2]} \\
&= 2(p+p')^\sigma \int_k^\Lambda \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2-m^2)^4} \\
&\quad + \int_k \frac{k_\mu k_\nu k_\rho (p'^2 - 2p'\cdot k)^2}{(k^2-m^2)^3[(p-k)^2-m^2][(p'-k)^2-m^2]} \\
&\quad + \int_k \frac{k_\mu k_\nu k_\rho (p^2 - 2p\cdot k)^2}{(k^2-m^2)^4[(p-k)^2-m^2]} \\
&\quad + \int_k \frac{k_\mu k_\nu k_\rho (p'^2 - 2p'\cdot k)(p^2-2p\cdot k)^2}{(k^2-m^2)^4[(p-k)^2-m^2]} \\
&= 2(p+p')^\sigma \int_k^\Lambda \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2-m^2)^4} + \text{non-ambiguous terms}\,,
\end{aligned} \tag{71}$$

which was expanded with the use of the identity (9) and where we discarded the terms with odd integrand in k. The infrared cutoff m^2 disappears in the end. The divergent integral can be written as a function of surface terms. Let us define

$$\alpha_2 g_{\mu\nu} \equiv \int_k^\Lambda \frac{g_{\mu\nu}}{(k^2-m^2)^2} - 4\int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)^3} \tag{72}$$

and

$$\alpha_3 g_{\{\mu\nu} g_{\alpha\beta\}} \equiv g_{\{\mu\nu} g_{\alpha\beta\}} \int_k^\Lambda \frac{1}{(k^2-m^2)^2} - 24 \int_k^\Lambda \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^4}\,. \tag{73}$$

The parameters α_2 and α_3 are surface terms. It can easily be shown that

$$\alpha_2 g_{\mu\nu} = \int_k^\Lambda \frac{\partial}{\partial k^\mu}\left(\frac{k_\nu}{(k^2-m^2)^2}\right) \tag{74}$$

and

$$\int_k^\Lambda \frac{\partial}{\partial k^\beta}\left[\frac{4k_\mu k_\nu k_\alpha}{(k^2-m^2)^3}\right] = g_{\{\mu\nu}g_{\alpha\beta\}}(\alpha_3-\alpha_2)\,. \tag{75}$$

These surface terms can be calculated by means of symmetric integration, with the substitutions $k_\mu k_\nu \to g_{\mu\nu}k^2/4$ and $k_\mu k_\nu k_\alpha k_\beta \to g_{\{\mu\nu}g_{\alpha\beta\}}k^4/24$. We obtain

$$\alpha_2 = -\frac{\mathrm{i}}{32\pi^2}\,, \quad \alpha_3 = \frac{5\mathrm{i}}{96\pi^2}\,. \tag{76}$$

Returning to the integral, we have

$$\begin{aligned} J_{\mu\nu\rho} &= -\frac{1}{12}(p+p')^\sigma g_{\{\mu\nu}g_{\rho\sigma\}}\alpha_3 + \ldots \\ &= -\frac{1}{12}(p+p')^\sigma g_{\{\mu\nu}g_{\rho\sigma\}}\frac{5i}{96\pi^2} + \text{non-ambiguous terms}\,. \end{aligned} \tag{77}$$

For the integral $I_\mu(p,p')$, one can write

$$\begin{aligned} I_\mu &- 2(p+p')^\nu \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)^3} + \ldots \\ &= \frac{1}{2}(p+p')^\nu g_{\mu\nu}\alpha_2 + \ldots \\ &= -\frac{\mathrm{i}}{64\pi^2}(p+p')^\nu g_{\mu\nu} + \text{non-ambiguous terms}\,. \end{aligned} \tag{78}$$

When the results of (77) and (78) are inserted into (70), and d is chosen to cancel these surface terms, it is found that $d=\frac{\mathrm{i}}{3}$, as expected.

The momentum-space version of (57), taking into account that $f=-g$ and $\mu=0$, is written

$$\hat{B}[k_\mu k_\nu] = \frac{1}{3}\left(p_\mu p_\nu - \frac{1}{4}g_{\mu\nu}p^2\right) I + \frac{1}{16\pi^2} f(p_\mu p_\nu - g_{\mu\nu}p^2)\,. \tag{79}$$

It is important to call the reader's attention to the fact that the decomposition in a traceless plus a trace part was not applied to this basic function. Instead, a differential equation was solved for $x \neq 0$ in order to find the term on $B[1]$. When the decomposition is performed, the traceless part is responsible for an ambiguity due to surface terms (ST), and a local term is added in order to take care of this problem. For the present case, there is also an ambiguity, but it is not due to the first term (on $B[1]$), which is already free from the ST. It emerges from the solution of the differential equation. So the constant f is not the counterterm to cancel the surface term. It is actually the local term that survives after the surface terms were eliminated. CDR assures it with the use of some consistency equations obtained with the help of the Leibniz rule. In the case of IR, the same result is achieved by explicit calculation, as long as ST are discarded. Let us expand it with the help of the identity (9) (m^2 will be set to zero in the final result):

$$\begin{aligned} \hat{B}[k_\mu k_\nu] = I_{\mu\nu} &= \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)[(p-k)^2-m^2]} \\ &= \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)}\left\{\frac{1}{(k^2-m^2)} - \frac{p^2-2p\cdot k}{(k^2-m^2)^2}\right. \\ &\quad \left. + \frac{(p^2-2p\cdot k)^2}{(k^2-m^2)^3} - \frac{(p^2-2p\cdot k)^3}{(k^2-m^2)^3[(p-k)^2-m^2]}\right\}\,. \end{aligned} \tag{80}$$

By discarding the terms with an odd integrand and remembering that we can use a parametrization so that the quadratic divergence is proportional to m^2, we obtain

$$\begin{aligned} I_{\mu\nu} = &-p^2\int_k^\Lambda \frac{k_\mu k_\nu}{(k^2-m^2)^3} + 4p^\alpha p^\beta \int_k^\Lambda \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^4} \\ &+ \text{finite}\,, \end{aligned} \tag{81}$$

in which we can use (72) and (73) to write

$$\begin{aligned} I_{\mu\nu} = &\frac{1}{3}\left(p_\mu p_\nu - \frac{1}{4}g_{\mu\nu}p^2\right) I_{\log}(m^2) \\ &+ \text{ST} + \text{finite terms}\,. \end{aligned} \tag{82}$$

The scale relation (16), the elimination of surface terms and the calculus of the finite part yields

$$\begin{aligned} I_{\mu\nu} = &\frac{1}{3}\left(p_\mu p_\nu - \frac{1}{4}g_{\mu\nu}p^2\right)\left\{I_{\log}(\lambda^2) - b\ln\left(-\frac{p^2}{\mathrm{e}^2\lambda^2}\right)\right\} \\ &- \frac{1}{18}b(p_\mu p_\nu - g_{\mu\nu}p^2)\,, \end{aligned} \tag{83}$$

where $b=\mathrm{i}/(4\pi)^2$. Then the constant f is found to be the same as the one fixed by CDR.

In the analysis carried out in this section, we have verified that all the rules and characteristics of constrained differential regularization can be mapped into the steps of implicit regularization. The only reason we have preferred to restrict ourselves, in the examples that we worked out, to the massless case is its simplicity, but all the features are present. The same procedure can be applied to the massive case, even when the problem involves particles with different masses. In this case, the expressions are greater and the whole process is more tedious, but there does not emerge any new feature. Besides, these two methods were applied to a large set of problems, massive and non-massive, always yielding equivalent results.

5 Concluding comments

The equivalence between constrained implicit regularization (CIR) and constrained differential renormalization (CDR) has been analyzed in this paper. The two methods have been tested, with positive and equivalent results, in many non-trivial situations, from the symmetry point of view. The physical dimension of the theory to be treated is not modified and, for this reason, these methods are candidates to be good tools in supersymmetric calculations. In the analysis carried out in this work it has been shown that each one of the rules of CDR, in position space, have its counterpart in momentum space, materialized in one of the rules of CIR. The relation has been shown to be one to one. The main characteristic of the two frameworks is the extension of properties of regular mathematical objects to the regularized ones. This is accomplished with the help of symmetry restoring counterterms. In practice it is very simple, as long as it is implemented by a set of rules.

The principles of CIR are successfully being applied at higher order calculations [43]. Differential renormalization at higher order has been used in various situations. We believe that the equivalence between these two frameworks at all orders could also be shown.

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